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ON THE RESIDUAL STRESSES IN THE VICINITY OF A CYLINDRICAL DISCONTINUITY IN A VISCOELASTOPLASTIC MATERIAL

A. A. Burenin, L. V. Kovtanyuk, and E. V. Murashkin

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The one-dimensional deformation of a material and subsequent unloading in the vicinity of a single cylindrical discontinuity is calculated using the theory of large viscoelastoplastic strains. Emphasis is on the formation of a residual stress field during the loading–unloading process and the effect of the viscous properties of the material on the level and distribution of these stresses. A comparison is performed with results of solution of the corresponding problem using the theory of large elastoplastic strains.

Key words: *finite strains, viscosity, plastic flow, residual stress.*

Introduction. The formation of a residual stress field in the vicinity of a cylindrical discontinuity has been previously considered for the model of large elastoplastic strains [1]. It turned out that the assumption of an ideal nature of plastic flow and allowance for only the elastic properties of the material during its deformation before plastic flow and during unloading are responsible for the adjustability of the defect to cyclic loads. In other words, irreversible strains near the defect are not accumulated with increase in the number of cycles, and residual stresses in its vicinity remain unchanged after each unloading. It is obvious that allowance for the viscous properties of the material during its irreversible deformation leads to the deceleration of plastic flow and, hence, to the development of a discontinuity. The rate of discontinuity development is the main factor that determines the fatigue strength of the article working under cyclic loads. The manifestation of viscous properties in the deformation stage preceding plastic flow or in the unloading stage is less obvious. Below, we consider exactly this case, assuming, as in [1], that the plastic flow is ideal. We note that the displacements of points of the medium being deformed in the vicinity of the discontinuity are commensurable with the defect size; therefore the assumption of small strains cannot be used. In the vicinity of the discontinuity they are always larger.

1. Basic Modeling Relations. One of the goals of the present study is to compare the results obtained with the results of solution of the problem in question for the model of an ideal elastoplastic medium. As noted above, this problem is considered in [1] using the model of large elastoplastic strains proposed in [2]. Therefore, the given mathematical model is chosen as the basis for the construction of further modeling relations. In [2], the splitting of the total Almansi strains d_{ij} into the reversible component e_{ij} and irreversible component p_{ij} is based on the requirement that the components of the latter vary during unloading in a similar manner as in the case of rigid body motion. Therefore, we define the components of the total Almansi strains using the equations of their transfer [3]:

$$\frac{dp_{ij}}{dt} = \varepsilon^p_{ij} - \varepsilon^p_{ik} p_{kj} - p_{ik}\varepsilon^p_{kj} + r_{ik}p_{kj} - p_{ik}r_{kj},$$

$$\frac{de_{ij}}{dt} = \varepsilon_{ij} - \varepsilon^p_{ij} - (1/2)\,(e_{ik}v_{k,j} + v_{k,i}e_{kj} - r_{ik}e_{kj} + e_{ik}r_{kj} - \varepsilon^p_{ik}e_{kj} - e_{ik}\varepsilon^p_{kj}), \tag{1.1}$$

$$2\varepsilon_{ij} = v_{i,j} + v_{j,i}, \qquad v_i = \frac{\partial u_i}{\partial t} + v_j u_{i,j}, \qquad r_{ij} = (1/2)\,(v_{i,j} - v_{j,i}) + F_{ij}(e_{st}, \varepsilon_{st}).$$

Institute of Automatics and Control Processes, Far East Division, Russian Academy of Sciences, Vladivostok 690041; burenin@iacp.dvo.ru. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 47, No. 2, pp. 110–119, March–April, 2006. Original article submitted May 30, 2005.

Relations (1.1) are written in Euler variables in rectangular Cartesian coordinates. Here u_i are the components of the displacement vector; the components of the skew tensor F_{ij} are not written and explicit expressions for them are given in [2]. The first relation in (1.1) can be treated as the definition of the plastic strain rate tensor ε_{ij}^p. This is the source tensor in the transfer equation for the tensor p_{ij}, and, hence, the relation is a definition of the objective derivative that links p_{ij} and ε_{ij}^p. It is possible to show (see [2]) that the components of the Almansi strain tensor d_{ij} are calculated by the relations

$$d_{ij} = e_{ij} + p_{ij} - (1/2)e_{ik}e_{kj} - e_{ik}p_{kj} - p_{ik}e_{kj} + e_{ik}p_{ks}e_{sj}. \tag{1.2}$$

Next we assume that the medium being deformed is incompressible. The relationship among the stress deviators, strains, and strain rates can be given by the simple linear relation

$$\begin{gathered} \tau_{ij} + \alpha \frac{D\tau_{ij}}{Dt} = 2\mu m_{ij} + 2\beta\varepsilon_{ij} \qquad \text{for} \quad p_{ij} \equiv 0, \\ \tau_{ij} + \alpha \frac{D\tau_{ij}}{Dt} = 2\mu m_{ij}^e + 2\beta\varepsilon_{ij} \qquad \text{for} \quad p_{ij} \neq 0, \\ \tau_{ij} = \sigma_{ij} - (1/3)\sigma_{kk}\delta_{ij} = \sigma_{ij} - p\delta_{ij}, \qquad m_{ij} = d_{ij} - (1/3)d_{kk}\delta_{ij}, \\ m_{ij}^e = e_{ij} - (1/2)e_{ik}e_{kj} - (1/3)e_{kk}\delta_{ij} + (1/6)e_{ks}e_{sk}\delta_{ij}. \end{gathered} \tag{1.3}$$

Here α, μ, and β are constants of the material and D/Dt is the Jaumann derivative. It should be noted that the first relation in (1.3) is the limiting one for the second as the irreversible strains tend to zero.

Applying the Mises maximum principle, we have the associated plastic-flow law

$$\varepsilon_{ij}^p = \lambda \frac{\partial f}{\partial \sigma_{ij}}, \qquad \lambda \geqslant 0. \tag{1.4}$$

Below as the plastic potential, we use the Tresca prism

$$f(\sigma_{ij}) = \max |\sigma_i - \sigma_j| - 2K = 0. \tag{1.5}$$

To close system (1.1)–(1.5) in the isothermal case considered, it suffices to write the usual kinematic relations and differential consequences of the conservation laws.

2. Formulation of the Problem. Initial Conditions of Plastic Flow. The microcrack modeling the discontinuity is considered long enough compared to its cross-sectional size. If the effect of the microcrack tips on the deformation of the material away from them is ignored, the deformation can be considered one-dimensional. To simplify the problem further, we assume that the defect is a cylindrical cavity of radius r_0 and the compression of the material is specified by the pressure on a cylindrical surface of initial radius $R_0 \gg r_0$:

$$\sigma_{rr}|_{r=R} = -p(t). \tag{2.1}$$

Here $R(t)$ is the current radius of the outer cylindrical surface and σ_{rr} is the stress tensor component in cylindrical coordinates (r, θ, z). The boundary surface of the defect is considered free:

$$\sigma_{rr}|_{r=s} = 0 \tag{2.2}$$

[$s(t)$ is the current radius of the cylindrical discontinuity]. As the function $p(t)$ increases, the material undergoes viscoelastic deformation until this function reaches the threshold value $p(t_0) = p_0$. At the same time t_0, the stress state on the boundary $r = s(t_0) = s_0$ reaches the loading surface (1.5) which is given by the equation

$$\sigma_{rr} - \sigma_{\theta\theta} = 2K. \tag{2.3}$$

Ignoring inertia effects, we use this state as the initial one for the subsequent process of plastic flow. The incompressibility state of the material is given by

$$(1 - u_{,r})\Big(1 - \frac{u}{r}\Big) = 1, \qquad u_{,r} = \frac{\partial u}{\partial r}, \tag{2.4}$$

where $u(r) = u_r$ is the single nonzero displacement component, which allows us to determine the kinematics of the medium with accuracy up to an arbitrary function of time $\varphi = \varphi(t)$ [$s(t)$ or $R(t)$]:

$$u = r - (r^2 + \varphi)^{1/2}, \qquad \varphi = R_0^2 - R^2 = r_0^2 - s^2,$$

$$2d_{rr} = 1 - \eta^{-1}, \qquad 2d_{\theta\theta} = 1 - \eta, \qquad \eta = 1 + r^{-2}(R_0^2 - R^2), \tag{2.5}$$

$$v = v_r = -\dot{\varphi}/2r.$$

Here the dot denotes the derivative with respect to time. With allowance for (2.5), the first equality (1.3) implies

$$\tau_{rr} + \alpha\Big(\frac{\partial \tau_{rr}}{\partial t} - \frac{\dot{\varphi}}{2r}\,\tau_{rr,r}\Big) = \frac{2\mu}{3}\Big(1 - \Big(1 + \frac{\varphi}{r^2}\Big)^{-1}\Big) + \mu\,\frac{\varphi}{3r^2} + \beta\,\frac{\dot{\varphi}}{r^2},$$

$$\tau_{\theta\theta} + \alpha\Big(\frac{\partial \tau_{\theta\theta}}{\partial t} - \frac{\dot{\varphi}}{2r}\,\tau_{\theta\theta,r}\Big) = -\frac{\mu}{3}\Big(1 - \Big(1 + \frac{\varphi}{r^2}\Big)^{-1}\Big) - 2\mu\,\frac{\varphi}{3r^2} - \beta\,\frac{\dot{\varphi}}{r^2}. \tag{2.6}$$

In (2.6), $(1+r^{-2}\varphi)^{-1}$ can be regarded as the sum of an infinitely decreasing geometric progression; therefore, τ_{rr} and $\tau_{\theta\theta}$ can be written as

$$\tau_{rr} = \sum_{n=1}^{\infty} \frac{a_n(t)}{n!r^{2n}}, \qquad \tau_{\theta\theta} = \sum_{n=1}^{\infty} \frac{b_n(t)}{n!r^{2n}}, \tag{2.7}$$

where $a_n(t)$ and $b_n(t)$ are unknown functions. Replacing the sum of a geometric progression in (2.6) by an infinite series and taking into account (2.7), we obtain

$$T(a_n) - \mu\,\frac{\varphi}{r^2} - \beta\,\frac{\dot{\varphi}}{r^2} + \frac{2\mu}{3}\,L(r,\varphi) = 0, \qquad T(b_n) + \mu\,\frac{\varphi}{r^2} + \beta\,\frac{\dot{\varphi}}{r^2} - \frac{\mu}{3}\,L(r,\varphi) = 0,$$

$$T(a_n) = \sum_{n=1}^{\infty} \frac{a_n + \alpha\dot{a}_n}{n!r^{2n}} + \alpha\dot{\varphi}\sum_{n=1}^{\infty} \frac{a_n}{(n-1)!r^{2n+2}}, \tag{2.8}$$

$$L(r,\varphi) = \sum_{n=2}^{\infty} (-1)^n \Big(\frac{\varphi}{r^2}\Big)^n.$$

The system of differential equations (2.6) for $\tau_{rr}(r,t)$, $\tau_{\theta\theta}(r,t)$, and $\varphi(t)$ is closed by the equation obtained from the equation of motion of the medium by integrating over r in the range $s \leqslant r \leqslant R$ using (2.5) and conditions (2.1) and (2.2). With allowance for (2.7), this equation can be written as

$$H(a_n, b_n, R, s) + M(\varphi, s, R) + p(t) = 0,$$

$$H(a_n, b_n, R, s) = \sum_{n=1}^{\infty} \frac{a_n - b_n}{2nn!}\Big(\frac{1}{R^{2n}} - \frac{1}{s^{2n}}\Big), \tag{2.9}$$

$$M(\varphi, s, R) = \frac{1}{2}\,\rho\Big(\ddot{\varphi}\ln(sR^{-1}) + \frac{1}{4}\Big(\frac{1}{R^2} - \frac{1}{s^2}\Big)\dot{\varphi}^2\Big).$$

Here ρ is the density of the medium.

Equating the coefficients of the same powers of r in (2.7) and (2.8) and using (2.9), we obtain the following infinite recursive system of ordinary differential equations for the functions of time $\varphi(t)$, $a_n(t)$, and $b_n(t)$:

$$\dot{a}_1 = \alpha^{-1}(-a_1 + \mu\varphi + \beta\dot{\varphi}),$$

$$\dot{b}_1 = \alpha^{-1}(-b_1 - \mu\varphi - \beta\dot{\varphi}),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \tag{2.10}$$

$$\dot{a}_n = \alpha^{-1}(-a_n - \alpha n(n-1)\dot{\varphi}a_{n-1} - (2\mu/3)(-1)^n n!\varphi^n),$$

$$\dot{b}_n = \alpha^{-1}(-b_n - \alpha n(n-1)\dot{\varphi}b_{n-1} + (\mu/3)(-1)^n n!\varphi^n).$$

Since the deformation begins from the free state, the initial conditions of the problem for the given system of ordinary differential equations are homogeneous: $\varphi(0) = \dot{\varphi}(0) = a_n(0) = b_n(0) = 0$. Truncation of series (2.7) to a finite number of terms makes it possible to solve system (2.10) numerically. Calculations show that despite the small sizes of the discontinuity, series (2.7) converge fairly rapidly. In numerical calculations, it suffices to take six terms of the series.

The problem described above is auxiliary. Calculations using the proposed scheme should be stopped when a region of plastic flow begins to form, which occurs, as noted above, at the time $t = t_0$, when the loading pressure reaches the threshold value $p = p_0$. At this time, condition (1.5) is satisfied on the boundary of the discontinuity $r = s_0$. In the notation adopted for the case considered, this condition is written as

$$\sum_{n=1}^{\infty} \frac{a_n - b_n}{n! s_0^{2n}} = 2K. \tag{2.11}$$

The radius s_0 at which plastic flow begins is calculated according to (2.11). The distributions of the stress–strain parameters calculated in such a manner are the initial conditions for the calculation of the parameters of the further deformation.

3. Plastic Flow. Let the loading force $p(t)$ continue to grow, so that

$$\sigma_{rr}|_{r=R(t)} = -p_0 - g(t), \qquad g(t_0) = 0, \qquad g(t) > 0 \quad \text{for} \quad t > t_0. \tag{3.1}$$

At $t > t_0$, the material in the region $s(t) \leqslant r \leqslant m(t)$ is in the plastic state. The unknown function $m(t)$ specifies the motion of the boundary of the plastic flow region. The equation of motion of the medium should be integrated separately in the plastic flow region and in the viscoelastic strain region. Using formula (2.7) for the stresses and integrating, we obtain

$$\sigma_{rr} = 2K \ln(sr^{-1}) + M(\varphi, s, r), \qquad \sigma_{\theta\theta} = \sigma_{rr} - 2K \tag{3.2}$$

in the plastic flow region and

$$\sigma_{rr} = H(a_n, b_n, r, R) + M(\varphi, R, r) - p_0 - g(t), \qquad \sigma_{\theta\theta} = \sigma_{rr} - \sum_{n=1}^{\infty} \frac{a_n - b_n}{n! r^{2n}} \tag{3.3}$$

in the viscoelastic strain region.

Relations (3.2) and (3.3) define the stress state in the material with accuracy up to the unknown functions of time $\varphi(t)$, $a_n(t)$, and $b_n(t)$. To calculate the latter and determine the function $m(t)$, which specifies the motion of the elastoplastic boundary, we use the conditions of equality of stresses (3.2) and (3.3) on this boundary [at $r = m(t)$]. As a result, we have the following equations for $\varphi(t)$ and $m(t)$:

$$\begin{gathered} H(a_n, b_n, m, R) + M(\varphi, R, s) - p_0 - g(t) - 2K \ln(sm^{-1}) = 0, \\ \dot{m} = m\Big(-\frac{2K}{\alpha} + \frac{\dot{a}_1 - \dot{b}_1}{m^2} + \frac{a_1 - b_1}{\alpha m^2} - \sum_{n=2}^{\infty} \frac{\varphi(a_{n-1} - b_{n-1})}{(n-2)! m^{2n}} + \frac{\mu}{3\alpha} L(m, \varphi)\Big)\Big(\sum_{n=1}^{\infty} \frac{2n(a_n - b_n)}{n! m^{2n}}\Big)^{-1}. \end{gathered} \tag{3.4}$$

Relations (3.4) need to be supplemented by system (2.10). Truncation of the resulting relation to a finite number of a_n and b_n allows the Cauchy problem for this system of ordinary differential equations to be studied numerically. It should be noted that in this case the initial conditions are the values of the functions obtained by solving the problem in Sec. 2 for the time $t = t_0$ (for $p = p_0$). Using the coefficients a_n and b_n of series (2.7) found by solving the indicated system of equations and the functions φ and m, we can construct the strain and stress fields at any time. The total strains are determined from the known displacements (2.5):

$$d_{rr} = \frac{1}{2} \frac{\varphi}{r^2 + \varphi}, \qquad d_{\theta\theta} = -\frac{\varphi}{2r^2}. \tag{3.5}$$

The strain components in the region of viscoelastic strain are calculated according to (1.2) in the form

$$e_{rr} = 1 - \sqrt{1 - 2d_{rr}}, \qquad e_{\theta\theta} = e_{rr}/(e_{rr} - 1). \tag{3.6}$$

To calculate the viscoelastic strain components in the plastic region, we use the second formula in (1.3), which in our case implies

$$2\mu(e_{rr} - e_{\theta\theta}) - \mu(e_{rr}^2 - e_{\theta\theta}^2) + 2\beta\dot{\varphi}/r^2 = 2K. \tag{3.7}$$

With allowance for (3.6), for the viscoelastic strain components we have the relations

$$e_{rr} = 1 - q^{1/2}, \qquad e_{\theta\theta} = 1 - q^{-1/2},$$
$$q = q_1 + \sqrt{q_1^2 + 1}, \qquad q_1 = -K/\mu + (\beta/\mu)\dot{\varphi}/r^2. \tag{3.8}$$

The plastic strains are calculated from the total [see (3.5)] and viscoelastic [see (3.8)] strains:

$$p_{rr} = (2d_{rr} + q - 1)/(2q), \qquad p_{\theta\theta} = q(2d_{\theta\theta} + q^{-1} - 1)/2. \tag{3.9}$$

Relations (3.8) and (3.9) define the distributions of the total strain components at any time in terms of the previously found function $\varphi(t)$. Data on these distributions are required to calculate the unloading process.

4. Unloading State. For the unloading of the material, we impose the following boundary conditions:

$$\sigma_{rr}|_{r=R(t)} = -p_1 + h(t), \qquad \sigma_{rr}|_{r=s(t)} = 0. \tag{4.1}$$

Here the constant p_1 is the value of the loading pressure attained during loading: $p_1 = -p_0 - g(t_1)$. The time $t = t_1$ is the moment when loading ceases and unloading begins. The monotonic function $h(t)$, defined for $t \geqslant t_1$, is such that $h(t_1) = 0$ and $h(t) > 0$. In the case where p_1 is large enough, the unloading process can initiate new plastic flow [4] because the stress state on the boundary of the defect reaches the loading surface (1.5) under tensile internal forces [$\sigma_{\theta\theta} = 2K$ at $r = s(t)$]. Let us consider this in a more general case. We assume that repeated plastic flow during unloading begins at the time $t = t_2$, when $R(t_2) = R_2$ and $s(t_2) = s_2$.

For the times $t_1 \leqslant t \leqslant t_2$, the following relations are valid:

$$u = r - (r^2 + \gamma)^{1/2}, \qquad \gamma = \varphi(t) = R_0^2 - R^2(t) = r_0^2 - s^2(t),$$
$$m^2(t) = m_1^2 - \gamma + \gamma_1, \qquad m_1 = m(t_1), \qquad \gamma_1 = \varphi(t_1). \tag{4.2}$$

In the region $m(t) \leqslant r \leqslant R(t)$, where there are no plastic strains, the stresses are calculated by formulas (3.3). Although during unloading in the time interval considered, a plastic region does not develop, the spatial coordinate of the boundary of the region $m(t)$ changes [$m(t) \neq m_1$], in contrast to the material coordinate, because of a change in the strain state. In the region $s \leqslant r \leqslant m$, the accumulated plastic strains are constant at each point of the medium (for each value of the material coordinate), but for the same value of the spatial coordinate, they are different. The spatial Eulerian coordinate r is linked to the material Lagrangian coordinate r_1 of a point of the medium that is fixed at the moment of the beginning of unloading by the relation $r^2 = r_1^2 - \gamma + \gamma_1$. Thus, plastic strains in the region $s \leqslant r \leqslant m$ should be calculated by relations (3.9) taking into account the given circumstance. Finally, elastic strains are determined from the known total and plastic strains:

$$e_{rr} = 1 - \sqrt{c^{-1}}, \qquad e_{\theta\theta} = 1 - \sqrt{c},$$
$$c = \frac{1}{q}\Big(1 + \frac{\gamma - \gamma_1}{r^2}\Big), \qquad q = -\frac{K}{\mu} + \sqrt{1 + \Big(\frac{K}{\mu}\Big)^2}. \tag{4.3}$$

As in the viscoelastic region, the stress deviator components in the plastic region are written as

$$\tau_{rr} = \sum_{n=0}^{\infty} \frac{z_n(t)}{n! r^{2n}}, \qquad \tau_{\theta\theta} = \sum_{n=0}^{\infty} \frac{w_n(t)}{n! r^{2n}}. \tag{4.4}$$

This representation allows us to integrate the equation of motion in the region $s \leqslant r \leqslant m$ and to obtain the following relations for the stress components:

$$\sigma_{rr} = H(z_n, w_n, r, s) + (z_0 - w_0) \ln (sr^{-1}) + M(\gamma, s, r),$$
$$\sigma_{\theta\theta} = \sigma_{rr} - \sum_{n=0}^{\infty} \frac{z_n - w_n}{n! r^{2n}}. \tag{4.5}$$

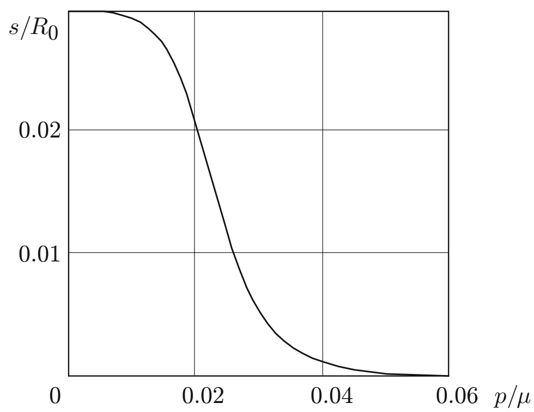

Fig. 1

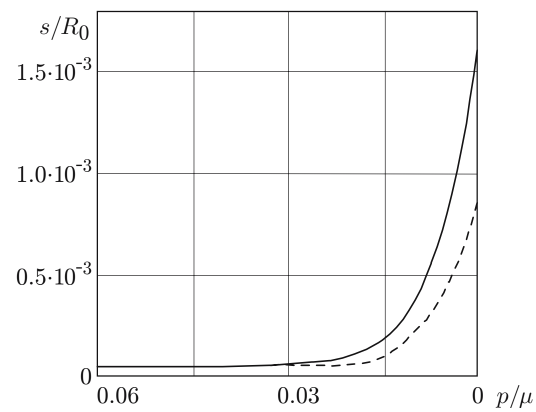

Fig. 2

Fig. 1. Variation in the boundary of the defect during loading.

Fig. 2. Variation in the boundary of the defect during unloading.

Expanding c^{-1} in (4.3) in a series in even negative powers of r and using (4.2), from the second relation (1.3) for the time interval considered, we obtain

$$
\begin{gathered}
z_0+\alpha\dot{z}_0+T(z_n)=\frac{2\mu}{3}\Big(\frac{1}{2}+\frac{1}{2q}+\frac{\gamma-\gamma_1}{2qr^2}-q\sum_{n=0}^{\infty}\frac{(\gamma_1-\gamma)^n}{r^{2n}}\Big)+\beta\,\frac{\dot{\gamma}}{r^2},\\
w_0+\alpha\dot{w}_0+T(w_n)=\frac{2\mu}{3}\Big(\frac{1}{2}-\frac{1}{q}-\frac{\gamma-\gamma_1}{qr^2}+\frac{q}{2}\sum_{n=0}^{\infty}\frac{(\gamma_1-\gamma)^n}{r^{2n}}\Big)-\beta\,\frac{\dot{\gamma}}{r^2}.
\end{gathered}
\tag{4.6}
$$

Comparing the coefficients at the same powers of r, we write the ordinary differential equations

$$
\begin{aligned}
\dot{z}_0&=\alpha^{-1}(-z_0+(2\mu/3)((1+1/q)/2-q)),\\
\dot{w}_0&=\alpha^{-1}(-w_0+(2\mu/3)((1+q)/2-1/q)),\\
\dot{z}_1&=\alpha^{-1}(-z_1+(2\mu/3)(\gamma-\gamma_1)(1/(2q)+q)+\beta\dot{\gamma}),\\
\dot{w}_1&=\alpha^{-1}(-w_1+(2\mu/3)(\gamma-\gamma_1)(1/q+q/2)-\beta\dot{\gamma}),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
\dot{z}_n&=\alpha^{-1}(-z_n-\alpha n(n-1)\dot{\gamma}z_{n-1}-(2\mu/3)qn!(\gamma_1-\gamma)^n),\\
\dot{w}_n&=\alpha^{-1}(-w_n-\alpha n(n-1)\dot{\gamma}w_{n-1}+(\mu/3)qn!(\gamma_1-\gamma)^n).
\end{aligned}
\tag{4.7}
$$

System (2.10) and (4.7) must be supplemented by the ordinary differential equation that follows from the condition of equality of the stresses σ_{rr} on the elastoplastic boundary $r=m$. In view of (4.5), this equation can be written as

$$
(z_0-w_0)\ln(sm^{-1})+H(z_n,w_n,m,s)-H(a_n,b_n,m,R)+M(\gamma,R,s)-p_1+h(t)=0. \tag{4.8}
$$

The closed system of equations (2.10), (4.7), and (4.8) subject to the constraints due to the finiteness of the number of terms in series (2.7) and (4.4) and the initial conditions at $t=t_1$ [$\gamma=\gamma_1$, $\dot{\gamma}=\dot{\gamma}(t_1)$, $a_n=a_n(t_1)$, $b_n=b_n(t_1)$, $z_n=z_n(t_1)=0$, and $w_n=w_n(t_1)=0$] is solved up to the time t_2 when the region of repeated plastic flow begins to spread from the boundary of the defect s_2. In the adopted notation, the condition of formation of this region (the Tresca plastic condition) under continuing unloading is written as

$$
\sum_{n=0}^{\infty}\frac{z_n-w_n}{n!s_2^{2n}}=-2K. \tag{4.9}
$$

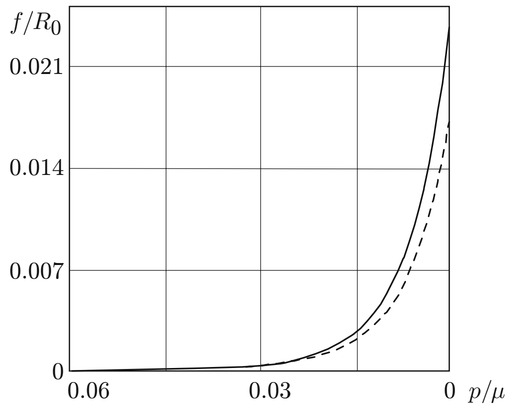

Fig. 3. Change in the boundaries of the region of repeated plastic flow.

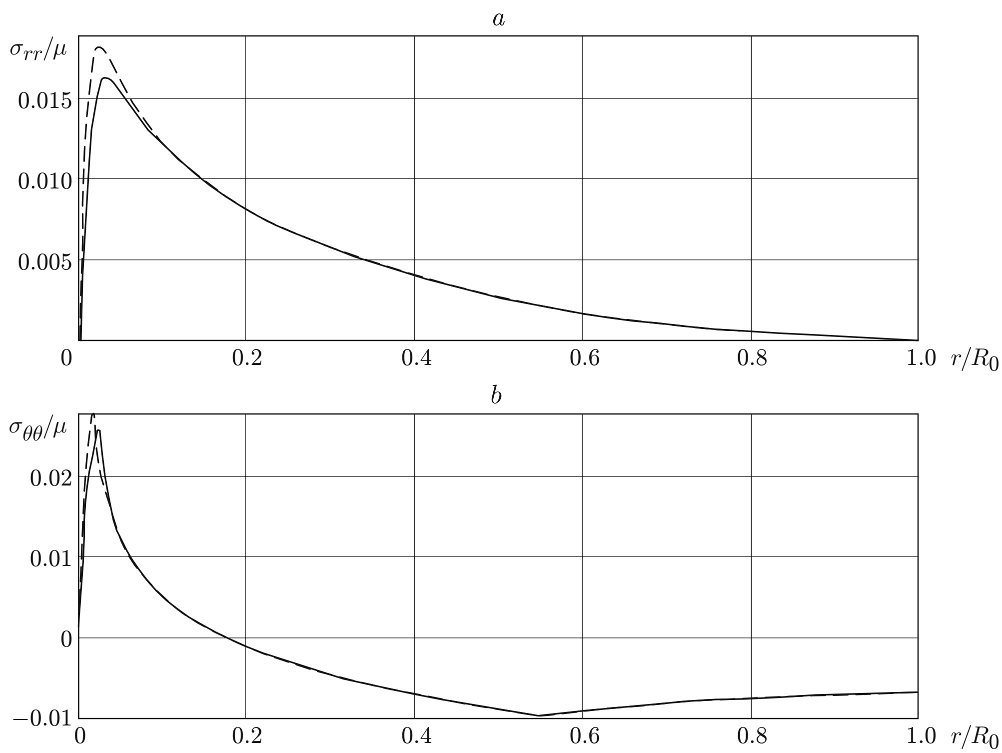

Fig. 4. Distribution of the residual radial stresses (a) and tangential stresses (b).

The values of the functions $\gamma_2 = \gamma(t_2)$, $\dot{\gamma}_2 = \dot{\gamma}(t_2)$, $a_n(t_2)$, $b_n(t_2)$, $z_n(t_2)$, and $w_n(t_2)$ become the initial conditions for the subsequent deformation process [under conditions (4.1)] with the region of repeated plastic flow developing during unloading. We assume that the plastic flow region is bounded by cylindrical surfaces: $s(t) \leqslant r \leqslant f(t)$ [$f(t)$ is the boundary of the region of repeated plastic flow]. The equation of motion of the medium should be integrated in three regions: $s \leqslant r \leqslant f$, $f \leqslant r \leqslant m$, and $m \leqslant r \leqslant R$. We note that $m \neq m_2 = m(t_2)$ although the plastic strains in the region $f \leqslant r \leqslant m$ are constant.

In the region $m \leqslant r \leqslant R$, plastic strains are absent and the stresses are defined by relations (3.3), in which the loading force $p_0 + g(t)$ should be replaced by its value for unloading $p_1 - h(t)$. In the region $f \leqslant r \leqslant m$, where the plastic strains do not change, the stresses can be calculated by relations (4.5):

$$\sigma_{rr} = (z_0 - w_0)\ln(mr^{-1}) + H(a_n, b_n, m, R) + H(z_n, w_n, r, m) + M(\gamma, R, m) - p_1 + h(t),$$
$$\sigma_{\theta\theta} = \sigma_{rr} - \sum_{n=0}^{\infty} \frac{z_n - w_n}{n! r^{2n}}. \tag{4.10}$$

In the region of repeated plastic flow $s \leqslant r \leqslant f$, integration of the equation of motion yields

$$\sigma_{rr} = -2K\ln(sr^{-1}) + M(\gamma, s, r),$$
$$\sigma_{\theta\theta} = \sigma_{rr} + 2K. \tag{4.11}$$

The condition of equality of the radial stresses σ_{rr} (4.10) and (4.11) on the boundary of the region of repeated plastic flow $r = f(t)$ implies the equation

$$(z_0 - w_0)\ln(mf^{-1}) + H(a_n, b_n, m, R) + H(z_n, w_n, f, m) + M(\gamma, R, s) - p_1 + h(t) + 2K\ln(sf^{-1}) = 0. \tag{4.12}$$

The motion of the boundary $r = f(t)$ of the region of repeated plastic flow is specified by the equation

$$\dot{f} = f\Big(\frac{2K}{\alpha} + \frac{\dot{z}_1 - \dot{w}_1}{f^2} + \frac{z_0 - w_0}{\alpha} + \frac{z_1 - w_1}{\alpha f^2} + \frac{2\eta\dot{\gamma}}{f^2}$$
$$- \sum_{n=2}^{\infty}\Big(\frac{\dot{\gamma}(z_{n-1} - w_{n-1})}{(n-2)! f^{2n}} + \frac{q\mu}{\alpha f^{2n}}(\gamma_1 - \gamma)^n\Big)\Big)\Big(\sum_{n=1}^{\infty}\frac{2n(z_n - w_n)}{n! f^{2n}}\Big)^{-1}. \tag{4.13}$$

The ordinary differential equations (4.12) and (4.13) close the infinite system of differential equations (2.10) and (4.7) for the functions γ, f, a_n, b_n, z_n, and w_n. Truncating series (2.7) and (4.4) to a finite number of terms and using the values of these functions at the time t_2 as the initial conditions, one can solve this problem numerically.

Let us consider some results of calculations for the following constants of the problem: $K/\mu = 0.003$, $r_0/R_0 = 0.03$, and $\eta/(\mu\alpha) = 3$ (shear modulus $\mu = 8.05 \cdot 10^{10}$). Figure 1 gives the variation in the boundary of the defect during loading. Figure 2 shows the difference in the defect size for unloading between the case of an ideal elastoplastic medium [1] (the solid curve in Figs. 2–4) and with allowance for viscosity during elastic deformation and during unloading (the dashed curve in Figs. 2–4). Allowance for the creep of the material leads to a decrease in the radius of the defect and in the size of the region of repeated plastic flow during unloading (Fig. 3). At the same time, the presumed significant decrease in the level of residual stresses due to their relaxation during unloading was not found numerically (Fig. 4). Although the numerical calculations performed in the present study and in [1] used different models, their results differ quantitatively only slightly.

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REFERENCES

1. A. A. Burenin, L. V. Kovtanyuk, and M. V. Polonik, "Formation of a one-dimensional field of residual stresses in the neighborhood of a cylindrical discontinuity in an elastoplastic medium," *Prikl. Mat. Mekh.*, **67**, No. 2, 316–325 (2003).
2. A. A. Burenin, G. I. Bykovtsev, and L. V. Kovtanyuk, "One simple model for an elastoplastic medium with finite strains," *Dokl. Ross. Akad. Nauk.*, **347**, No. 2, 199–201 (1996).
3. V. P. Myasnikov, "The equations of motion of elastoplastic materials with large strains," *Vest. Dalnevost. Otd. Ross. Akad. Nauk.*, **4**, 8–13 (1996).
4. A. A. Burenin, L. V. Kovtanyuk, and M. V. Polonik, "Possibility of repeated plastic flow during general unloading of an elastoplastic medium," *Dokl. Ross. Akad. Nauk.*, **375**, No. 6, 767–769 (2000).